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Existence of weighted pseudo anti-periodic solutions to some neutral differential equations with piecewise constant argument *

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Abstract: By means of weighted pseudo anti-periodic solutions of relevant difference equations, the existence for weighted pseudo anti-periodic solutions of differential equations with piecewise constant argument is studied. The conditions of existence and uniqueness for the weighted pseudo anti-periodic solutions are presented.

Key words: pseudo anti-periodic solutions; pseudo anti-periodic sequences; neutral delay equation; piecewise constant argument
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具有分段常变量的中立型微分方程加权伪反周期解的存在性

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摘　要：通过构造差分方程的加权伪反周期解，研究了一类含分段常变量中立型微分方程的加权伪反周期解的存在性，给出了所论方程的加权伪反周期解的存在唯一性条件。

关键词：伪反周期解；伪反周期序列；中立型时滞方程；分段常变量

In this paper we consider the following first order neutral delay differential equations with piecewise constant argument of the forms

$$[x(t)+px(t-1)]' = a_0x([t])+a_1x([t-1])+f(t) \tag{1}$$

$$[x(t)+px(t-1)]' = a_0x([t])+a_1x([t-1])+g(t,x(t),x([t])) \tag{2}$$

where $p(\neq 0), a_0, a_1$ are constants, $[\cdot]$ denotes the greatest integer function. To study the existence of weighted pseudo ω-anti-periodic solutions to Eqs. (1) and (2), we will assume that the following assumptions hold:

(H1) $f:\mathbf{R}\to\mathbf{R}$ is weighted pseudo ω-anti-periodic function.

(H2) $g:\mathbf{R}\times\mathbf{R}^2\to\mathbf{R}$ is jointly continuous and satisfies $g(t+\omega,x,y)=-g(t,x,y)$ for all $t\in\mathbf{R}$ and $(x,y)\in\mathbf{R}^2$. Moreover, the function g is uniformly Lipschitz with respect to x, y in the following sense: there exists $\eta>0$ such that

$$|g(t,x_1,y_1)-g(t,x_2,y_2)| \leqslant \eta[|x_1-x_2|+|y_1-y_2|] \tag{3}$$

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for all $(x_i, y_i) \in \mathbf{R}^2$, $i = 1, 2$ and $t \in \mathbf{R}$.

A function $x: \mathbf{R} \to \mathbf{R}$ is called a solution of Eq. (1) if the following conditions are satisfied:

(i) x is continuous on $\mathbf{R}$;

(ii) the derivative of $x(t) + px(t-1)$ exists on $\mathbf{R}$ except possibly at the points $t = n$, $n \in \mathbf{Z}$, where one-sided derivatives exist;

(iii) x satisfies Eq. (1) on each interval $(n, n+1)$, with integer $n \in \mathbf{Z}$.

The existence of anti-periodic solutions to differential equations is an attractive topic in the qualitative theory of differential equations due to its applications in control theory or engineering and others, see [1-4] and references therein. Motivated by the study of existence of pseudo almost periodic solutions, and weighted pseudo almost solution to differential equations[5-7], Al-Islam[8] et al. introduced the weighted pseudo anti-periodic functions, which is a natural generalization of the classical pseudo almost periodic functions, and has been used in the investigation of a certain non-autonomous second-order abstract differential equation.

Differential equations with piecewise constant arguments are usually referred to as a hybrid system (a combination of continuous and discrete). These equations have the structure of continuous dynamical systems within intervals and the solutions are continuous, and so combine properties of both differential and difference equations. The equations are thus similar in structure to those found in certain sequential-continuous models of disease dynamics as treated by by Busenberg and Cooke [9]. Therefore, there are many papers concerning the differential equations with piecewise constant argument (see [10-19] and the references therein).

We note that there is no results on the weighted pseudo anti-periodic solution for Eq. (1) (or (2)) still now. The main purpose of this work is to establish an existence and uniqueness result of weighted pseudo anti-periodic solutions of Eqs. (1) and (2).

1 Preliminary definitions and lemmas

For the sake of convenience, we now state some of the preliminary definitions and lemmas. we always denote by $BC(\mathbf{R}, \mathbf{R})$ the space of bounded continuous functions $u: \mathbf{R} \to \mathbf{R}$, $C(\mathbf{R}, \mathbf{R})$ the space of continuous functions $u: \mathbf{R} \to \mathbf{R}$, and denote by $|\cdot|$ the Euclidean norm.

Definition 1 A function $f \in C(\mathbf{R},\mathbf{R})$ is said to be ω-anti-periodic function for some $\omega > 0$, if $f(t+\omega) = -f(t)$ for all $t \in \mathbf{R}$. The least positive ω with this property is called the anti-period of f. Denote by $AP_\omega(\mathbf{R})$ the set of all such functions.

Proposition 1 If $f(t)$ is an ω-anti-periodic function, then $f(t)$ is also $(2\omega+1)$-anti-periodic and 2ω-periodic.

Let U be the collection of functions (weights) $\rho: \mathbf{R} \to (0, +\infty)$, which are locally integrable over $\mathbf{R}$. If $\rho \in U$, we set

$$\mu(T,\rho) := \int_{-T}^{T} \rho(t)\,\mathrm{d}t, T > 0,$$

$$U_\infty := \{\rho \in U : \lim_{T\to\infty} \mu(T,\rho) = \infty\}$$

and

$$U_B := \{\rho \in U_\infty : \rho \text{ is bounded with } \inf_{t\in\mathbf{R}} \rho(t) > 0\}$$

Obviously, $U_B \subset U_\infty \subset U$, with strict inclusions.

Let $\rho_1, \rho_2 \in U_\infty$, ρ_1 is said to be equivalent to ρ_2, denoting this as $\rho_1 \prec \rho_2$, if $\rho_1/\rho_2 \in U_B$. Then $\prec$ is a binary equivalence relation on U_∞ (see [7]). Let $\rho \in U_\infty$, $c \in \mathbf{R}$, define ρ_c by $\rho_c(t) = \rho(t+c)$ for $t \in \mathbf{R}$. We denote

$$U_T := \{\rho \in U_\infty : \rho \prec \rho_c \text{ for each } c \in \mathbf{R}\}$$

It is easy to see that U_T contains plenty of weights, say, 1, e^t, $1 + 1/(1+t^2)$, $1 + |t|^n$ with $n \in$ N, etc.

For $\rho \in U_\infty$, the weighted ergodic space $PAP_0(\mathbf{R},\rho)$ is defined by

$$PAP_0(\mathbf{R},\rho) := \{f \in BC(\mathbf{R},\mathbf{R}) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} |f(t)| \rho(t)\,\mathrm{d}t = 0\}$$

Lemma 1[14] $PAP_0(\mathbf{R},\rho)$ with $\rho \in U_T$ is translation invariant, i. e. $\varphi \in PAP_0(\mathbf{R},\rho)$ and $s \in \mathbf{R}$ imply that $\varphi(\cdot - s) \in PAP_0(\mathbf{R},\rho)$.

Definition 2 [7] Let $\rho \in U_\infty$. A function $f \in BC(\mathbf{R}, \mathbf{R})$ is called weighted ω-anti-periodic function (or ρ-pseudo ω-anti-periodic function) for some $\omega > 0$, if f can be written as $f = f^{ap} + f^e$, where $f^{ap} \in AP_\omega(\mathbf{R})$, and $f^e \in PAP_0(\mathbf{R},\rho)$. f^{ap} and f^e are called the ω-anti-periodic component and the weighted ergodic perturbation, respectively, of the function f. Denote by $PAP_\omega(\mathbf{R},\rho)$ the set of all such functions.

Definition 3 Let $\rho \in U_\infty$. A function $g \in BC(\mathbf{R}\times\mathbf{R})$ is called weighted pseudo ω-anti-periodic function (or ρ-pseudo ω-anti-periodic function) in t uniformly on $\mathbf{R}^2$, if g can be written as $g = g^{ap} + g^e$, where g^{ap} is ω-anti-periodic in t uniformly for $\mathbf{R}^2$, and for any compact set $W \subset \mathbf{R}^2$, g^e is continuous, bounded and satisfies

$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|g^e(t,x,y)|\rho(t)\mathrm{d}t = 0$$

uniformly in $(x,y) \in W$, g^{ap} and g^e are called the ω-anti-periodic component and the weighted ergodic perturbation, respectively, of the function g. Denote by $PAP(\mathbf{R}\times\mathbf{R},\mathbf{R},\rho)$ the set of all such functions.

Definition 4　A sequence $x:\mathbf{Z}\to\mathbf{R}$, denoted by $\{x(n)\}$, is called a ω-anti-periodic sequence if $x(n+\omega) = -x(n)$ for all $n\in\mathbf{Z}$. We denote the set of all such sequences by $AP\omega\ S(\mathbf{R})$.

Let U_s denote the collection of sequences (weights) $Q:\mathbf{Z}\to(0,+\infty)$. For $Q\in U_s$ and $T\in\mathbf{Z}^+ = \{n\in\mathbf{Z}: n\geqslant 0\}$, set

$$\mu_s(T,Q) = \sum_{n=-T}^{T}Q(n)$$

Denote

$$U_{s\infty} :=\{Q\in U_s:\lim_{N\to\infty}\mu_s(T,Q) = \infty\}$$

and

$$U_{sB} :=\{Q\in U_{s\infty}: Q \text{ is bounded with } \inf_{n\in\mathbf{Z}}Q(n) > 0\}$$

Let $Q_1, Q_2\in U_{s\infty}$, Q_1 is said to be equivalent to Q_2, denoting this as $Q_1 \prec Q_2$, if $\{Q_1(n)/Q_2(n)\}_{n\in z}\in U_{sB}$. Then it is easy to see that $\prec$ is a binary equivalence relation on $U_{s\infty}$. Let $Q\in U_{s\infty}$, $k\in\mathbf{Z}$, define Q_k by $Q_k(n) = Q(n+k)$ for $n\in\mathbf{Z}$. We denote

$$U_{sT} :=\{Q\in U_{s\infty}: Q\prec Q_k \text{ for each } k\in\mathbf{Z}\}$$

For $Q\in U_{s\infty}$, the weighted ergodic sequences space $PAP_0S(\mathbf{R},Q)$ is defined by

$$PAP_0S(\mathbf{R},Q) :=\left\{x: x \text{ is bounded with } \lim_{T\to\infty}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|x(n)|Q(n) > 0\right\}$$

Definition 5　Let $Q\in U_{s\infty}$. A sequence $x:\mathbf{Z}\to\mathbf{R}$, is called a weighted pseudo ω-anti-periodic sequence (or Q-pseudo ω-anti- periodic sequence) if x can be written as $x(n) = x^{ap}(n) + x^e(n)$, $n\in\mathbf{Z}$ where $x^{ap}\in AP_\omega S(\mathbf{R})$, and $x^e\in PAP_0S(\mathbf{R},Q)$. x^{ap} and x^e are called the ω-anti-periodic component and the weighted ergodic perturbation, respectively, of the sequence x. Denote the set of all such sequences by $PAP_\omega S(\mathbf{R},Q)$.

Proposition 2　If $f\in AP_\omega(\mathbf{R})$, $\omega\in\mathbf{Z}^+$, then $\{f(n)\}_{n\in\mathbf{Z}}\in AP_\omega S(\mathbf{R})$.

Lemma 2　If $f\in AP_\omega(\mathbf{R})$, $\omega\in\mathbf{Z}^+$, let $h_n = \int_n^{n+1}f(s)\mathrm{d}s$, then $\{h_n\}_{n\in\mathbf{Z}}\in AP_\omega S(\mathbf{R})$.

Proof　Since $f(t)$ is an ω-anti-periodic function, then for all $t\in\mathbf{R}$, we have $f(t+\omega)+f(t) = 0$ and

$$h_{n+\omega}+h_n = \int_{n+\omega}^{n+\omega+1}f(s)\mathrm{d}s + \int_n^{n+1}f(s)\mathrm{d}s = \int_n^{n+1}[f(s+\omega)+f(s)]\mathrm{d}s = 0$$

From definition, it follows that $\{h_n\}_{n\in\mathbf{Z}}$ is an ω-anti-periodic sequence. This completes the proof of Lemma 2.

Lemma 3　Let $\rho\in U_T$, and denote

$$Q(n) = \int_n^{n+1}\rho(t)\mathrm{d}t, \quad \text{for } n\in\mathbf{Z} \tag{4}$$

Then $Q\in U_{sT}$. Moreover, given $c\in\mathbf{R}$, there exist positive constants C_1, C_2 such that, for sufficiently large T,

$$C_1\mu(T+c,\rho)\leqslant\mu_s([T],Q)\leqslant C_2\mu(T+c,\rho) \tag{5}$$

Proof Without loss of generality, we assume that $c\geqslant 0$. Since $\rho\in U_T$, there exists $M>0$ such that $\rho_{c+1}(t)\leqslant M\rho(t)$ and $\rho_{-(c+1)}(t)\leqslant M\rho(t)$ for $t\in\mathbf{R}$ and

$$\mu(T-1,\rho)\leqslant\mu_s([T],Q) = \int_{-[T]}^{[T]+1}\rho(t)\mathrm{d}t\leqslant\mu(T+1+c,\rho) \tag{6}$$

Notice that

$$\mu(T+c,\rho) = \int_{-T-c}^{T+c}\rho(t)\mathrm{d}t = \int_{-T-2c-1}^{T-1}\rho_{c+1}(t)\mathrm{d}t = \int_{-T+1}^{T-1}\rho_{c+1}(t)\mathrm{d}t + \int_{-T-2c-1}^{-T+1}\rho_{c+1}(t)\mathrm{d}t = \int_{-T+1}^{T-1}\rho_{c+1}(t)\mathrm{d}t + \int_{-T+1}^{-T+2c+3}\rho_{-(c+1)}(t)\mathrm{d}t$$

For $T>c+2$, i. e., $-T+2c+3<T-1$, we then that

$$\mu(T+c,\rho)\leqslant\int_{-T+1}^{T-1}M\rho(t)\mathrm{d}t + \int_{-T+1}^{T-1}M\rho(t)\mathrm{d}t = 2M\mu(T-1,\rho) \tag{7}$$

Similarly, we can prove that there exists $M'>0$ such that, for T large enough,

$$\mu(T+1+c,\rho)\leqslant M'\mu(T+c,\rho) \tag{8}$$

Thus by (6) – (8), we have

$$\frac{1}{2M}\mu(T+c,\rho)\leqslant\mu_s([T],Q)\leqslant M'\mu(T+c,\rho)$$

for T large enough. This leads to (5), and from which we can get easily that $Q\in U_{sT}$. The proof is complete.

Proposition 3　$PAP_0S(\mathbf{R},Q)$ with $Q\in U_{sT}$ is translation invariant.

Proof Let $x \in PAP_0S(\mathbf{R},Q)$. Without loss of generality, we assume that $k > 0$. Than there exists $M>0$ such that $Q_k(n)/Q(n) < M$ for $n \in \mathbf{Z}$ since $Q \in U_{sT}$. Let $\rho(t) = Q(n)$ for $t \in [n, n+1)$, $n \in \mathbf{Z}$. Then $\rho \in U_T$ and $Q(n) = \int_n^{n+1}\rho(t)\,\mathrm{d}t$ for $n \in \mathbf{Z}$. Now applying Lemma 3, we can get that

$$\lim_{T\to\infty}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|x(n-k)|Q(n) \leqslant \lim_{T\to\infty}\frac{1}{\mu_s(T,Q)}\sum_{n=-(T+k)}^{T+k}|x(n)|Q_k(n) \leqslant \lim_{T\to\infty}\frac{\mu_s(T+k,Q)}{\mu_s(T,Q)}\cdot\frac{1}{\mu_s(T+k,Q)}\cdot\sum_{n=-(T+k)}^{-T+k}|x(n)|MQ(n) = 0$$

This implies that $\{x(n-k)\}_{n\in\mathbf{Z}} \in PAP_0S(\mathbf{R},Q)$. The proof is complete.

Lemma 4 Assume $f \in PAP_\omega S(\mathbf{R},\rho)$, $\omega \in \mathbf{Z}^+$, let $h_n = \int_n^{n+1} f(s)\,\mathrm{d}s$, then $\{h_n\}_{n\in\mathbf{Z}} \in PAP_\omega S(\mathbf{R},\rho)$.

Proof Let $f = f^{ap}+f^e$, where f^{ap} is ω-anti-periodic, and f^e is continuous, bounded and satisfies

$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|f^e(t)|\rho(t)\,\mathrm{d}t = 0$$

It is clear that $|h_n| \leqslant \|f\|$ for $n \in \mathbf{Z}$ and

$$h_n^{ap} = \int_n^{n+1} f^{ap}(t)\,\mathrm{d}t$$

is ω-anti-periodic. Let

$$h_n = h_n - h_n^{ap} = \int_n^{n+1} f^e(t)\,\mathrm{d}t$$

For $T \in \mathbf{Z}^+$, we get

$$\sum_{n=-T}^{T}|h_n^e|Q(n) \leqslant \sum_{n=-T}^{T}\int_n^{n+1}\int_n^{n+1}|f^e(s)|\,\mathrm{d}s\rho(t)\,\mathrm{d}t = \sum_{n=-T}^{T}\int_n^{n+1}\int_{n-t}^{n+1-t}|f^e(s+t)|\rho(t)\,\mathrm{d}t\mathrm{d}s \leqslant \sum_{n=-T}^{T}\int_{-1}^{1}\int_n^{n+1}|f^e(s+t)|\rho(t)\,\mathrm{d}t\mathrm{d}s \leqslant \int_{-1}^{1}\int_{-T-1}^{T+1}|f^e(s+t)|\rho(t)\,\mathrm{d}t\mathrm{d}s$$

For $T \in \mathbf{Z}^+$, $s \in [1, 1]$, let

$$\Phi_T(s) = \frac{1}{\mu(T+1,\rho)}\int_{-T-1}^{T+1}|f^e(s+t)|\rho(t)\,\mathrm{d}t$$

Then $|\Phi_T(s)| = \|f^e\|$. From Lemma 1, we get $\lim_{T\to\infty}\Phi_T(s) = 0$ for each $s \in [-1,1]$. Now by Lebesgue dominated convergence theorem and Lemma 3, we obtain

$$\lim_{T\to\infty}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|h_n^e|Q(n) \leqslant \lim_{T\to\infty}\frac{\mu(T+1,\rho)}{\mu_s(T,Q)}\int_{-1}^{1}\Phi_T(s)\,\mathrm{d}s = 0$$

That is $h_n^e \in PAP_0S(\mathbf{R},Q)$, and $\{h_n\}_{n\in\mathbf{Z}} \in PAP_\omega S(\mathbf{R},\rho)$. The proof is complete.

Lemma 5[11] Let $x:\mathbf{R}\to\mathbf{R}$ is a continuous function, and $w(t) = x(t) + px(t-1)$. then

$$|x(t)| \leqslant \mathrm{e}^{-a(t-t_0)}\sup_{-1\leqslant\theta\leqslant 0}|x(t_0+\theta)| + b\sup_{t_0\leqslant u\leqslant t}|w(u)|,\quad t \geqslant t_0$$

Where $|p| < 1$, $a = \log(1/|p|)$, $b = 1/(1-|p|)$, or

$$|x(t)| \leqslant \mathrm{e}^{\log|p|(t-t_0)}\sup_{0\leqslant\theta\leqslant 1}|x(t_0+\theta)| + b\sup_{t\leqslant u\leqslant t_0}|w(u+1)|,\quad t \leqslant t_0$$

Where $|p| > 1$, $b = 1/(|p|-1)$.

2 Main results

Now, we can formulate our main theorems.

Theorem 1 Suppose that

$$\begin{cases} p \neq 1 + \dfrac{a_0 - a_1}{2}, |p| < 1, \\ a_0 + a_1 \neq 0, \\ (p-1-a_0)^2 + 4(p+a_1) \neq 0 \end{cases} \tag{9}$$

Then for any $f \in PAP_\omega S(\mathbf{R},\rho)$, the following results hold:

(i) If $\omega = n_0 \in \mathbf{Z}^+$, Eq. (5) has a unique ρ-pseudo ω-anti-periodic bounded solution.

(ii) If $\omega = \dfrac{n_0}{m_0}$, $n_0, m_0 \in \mathbf{Z}^+$ and n_0, m_0 are mutually prime, Eq. (5) has a unique ρ-pseudo-$m\omega$-anti- periodic bounded solution, where $m = m_0$ if m_0 is odd, or $m = 2m_0+1$ if m_0 is even.

Theorem 2 Suppose that conditions (H_2) and (9) hold. Then there exists $\eta^* > 0$, such that if $\eta < \eta^*$, that following results hold:

(i) If $\omega = n_0 \in \mathbf{Z}^+$, Eq. (5) has a unique ρ-pseudo ω-anti-periodic bounded solution.

(ii) If $\omega = \dfrac{n_0}{m_0}$, $n_0, m_0 \in \mathbf{Z}^+$ and n_0, m_0 are mutually prime, Eq. (5) has a unique ρ-pseudo-$m\omega$-anti-periodic bounded solution, where $m = m_0$ if m_0 is odd, or $m = 2m_0+1$ if m_0 is even.

3 Proofs of theorems

Proof of Theorem 1 (i) Let $x(t)$ be a solution of Eq. (1) on $\mathbf{R}$, integrating (1) from n to t, we have that for $n \leqslant t < n+1$,

$$[x(t)+px(t-1)]=[x(n)+px(n-1)]+[a_0x(n)+a_1x(n-1)](t-n)+\int_n^t f(s)\mathrm{d}s \tag{10}$$

In view of the continuity of a solution at a point, we obtain that for $t \to (n+1)-0$,

$$x(n+1)+(p-1-a_0)x(n)+(-p-a_1)x(n-1)=h_n \tag{11}$$

where $h_n=\int_n^{n+i} f(s)\mathrm{d}s$.

The corresponding homogeneous equation of Eq. (11) is

$$x(n+1)+(p-1-a_0)x(n)+(-p-a_1)x(n-1)=0 \tag{12}$$

Following [10], we seek the particular solutions as $x(n)=\lambda^n$ for homogeneous difference equation (12), then we have the following characteristic equation of (12):

$$\lambda^2+(p-1-a_0)\lambda+(-p-a_1)=0 \tag{13}$$

Eq. (13) has two nontrivial solutions

$$\lambda_{1,2}=\frac{-(p-1-a_0)}{2}\pm\frac{\sqrt{(p-1-a_0)^2-4(-p-a_0)}}{2}$$

In view of (9), we have that $|\lambda_{1,2}|\neq 1$ and $\lambda_1\neq\lambda_2$, then

$$\{x(n)\}_{n\in\mathbf{Z}}=\{k_1\lambda_1^n+k_2\lambda_2^n\}_{n\in\mathbf{Z}} \tag{14}$$

is the general solutions of Eq. (12), where k_1, k_2 are any constants.

We define a sequence $\{c_n\}$ by

$$c_n=\begin{cases} k_1\sum_{m\leqslant n-1}\lambda_1^{n-(m+1)}h_m+k_2\sum_{m\leqslant n-1}\lambda_2^{n-(m+1)}h_m, & |\lambda_1|<1,|\lambda_2|<1, \\ k_1\sum_{m\leqslant n-1}\lambda_1^{n-(m+1)}h_m+k_2\sum_{m\geqslant n}\lambda_2^{n-(m+1)}h_m, & |\lambda_1|<1,|\lambda_2|>1, \\ k_1\sum_{m\geqslant n}\lambda_1^{n-(m+1)}h_m+k_2\sum_{m\leqslant n-1}\lambda_2^{n-(m+1)}h_m, & |\lambda_1|>1,|\lambda_2|<1, \\ k_1\sum_{m\geqslant n}\lambda_1^{n-(m+1)}h_m+k_2\sum_{m\geqslant n}\lambda_2^{n-(m+1)}h_m, & |\lambda_1|>1,|\lambda_2|>1 \end{cases} \tag{15}$$

where k_1, k_2 will be determined later. We put Eq. (15) into Eq. (11) and compare the coefficients of h_n's.

For $|\lambda_1|<1$, $|\lambda_2|<1$, we obtain a linear system in k_1 and k_2

$$\begin{cases} k_1\lambda_1+k_2\lambda_2=1+a_0-p, \\ k_1+k_2=1 \end{cases} \tag{16}$$

Solving system (16), we have

$$\begin{cases} k_1^0=\dfrac{1+a_0-p-\lambda_2}{\lambda_1-\lambda_2}, \\ k_2^0=\dfrac{1+a_0-p-\lambda_1}{\lambda_1-\lambda_2} \end{cases}$$

So, the sequence $\{\hat{c}_n\}$, defined by

$$\hat{c}_n=k_1^0\sum_{m\leqslant n-1}\lambda_1^{n-(m+1)}h_m+k_2^0\sum_{m\leqslant n-1}\lambda_2^{n-(m+1)}h_m \tag{17}$$

is a solution of the difference equation (11).

For other cases we can similarly write out the expression for the solution of Eq. (11).

(ii) Since $f\in PAP_\omega S(\mathbf{R},\rho)$, it follows from Lemma 4 that $\{h_n\}_{n\in\mathbf{Z}}\in PAP_\omega S(\mathbf{R},Q)$, so that $\{h_n\}$ can be written as a sum

$$h_n=h_n^{ap}+h_n^e$$

where $\{h_n^{ap}\}$ is a ω-anti-periodic sequence, and $\{h_n^e\}$ is bounded and satisfed

$$\lim_{N\to\infty}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|h_n^e|Q(n)=0$$

It is easy to see that

$$\hat{c}_n^e=k_1^0\sum_{m\leqslant n-1}\lambda_1^{n-(m+1)}h_m^e+k_2^0\sum_{m\leqslant n-1}\lambda_2^{n-(m+1)}h_m^e, n\in\mathbf{Z}$$

is bounded. In order to show the ρ-pseudo ω- anti-periodic periodicity of $\{\hat{c}_n\}$, it suffices to prove

$$\lim_{T\to\infty}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|\hat{c}_n^e|Q(n)=0$$

Indeed, it is easy to see that for $T\in\mathbf{Z}^+$,

$$\begin{aligned}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|\hat{c}_n^e|Q(n)={}&\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}\sum_{m=-\infty}^{n-1}|\lambda_1|^{n-m-1}\cdot|h_m^e|Q(n)+\\&\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}\sum_{m=-\infty}^{n-1}|\lambda_2|^{n-m-1}|h_m^e|Q(n)=\\&\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}\sum_{m=0}^{\infty}|\lambda_1|^{m}|h_{n-1-m}^e|Q(n)+\\&\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}\sum_{m=0}^{\infty}|\lambda_2|^{m}|h_{n-1-m}^e|Q(n)=\\&\sum_{m=0}^{\infty}|\lambda_1|^{m}\frac{1}{\mu_s(T,Q)}\sum_{n=-T}^{T}|h_{n-1-m}^e|Q(n)+\end{aligned}$$

$$\sum_{m=0}^{\infty} |\lambda_2|^m \frac{1}{\mu_s(T,Q)} \sum_{n=-T}^{T} |h_{n-1-m}^e| Q(n)$$

For $m \in \mathbf{Z}^+$, let

$$\Phi_T(m) = \frac{1}{\mu_s(T,Q)} \sum_{n=-T}^{T} |h_{n-1-m}^e| Q(n)$$

From Proposition 1, we get

$$\lim_{T\to\infty} \Phi_T(m) = 0,$$

$$\Phi_T(m) \leqslant \sup_{n\in\mathbf{Z}} |h_n^e| = M_1, \text{ for } m \in \mathbf{Z}^+ \quad (18)$$

Given $\varepsilon > 0$, it is clear that there exists an integer $K > 0$ such that

$$\sum_{m=K+1}^{\infty} |\lambda_1|^m < \varepsilon \text{ and } \sum_{m=K+1}^{\infty} |\lambda_2|^m < \varepsilon \quad (19)$$

Then by (18), there exists $T_0 > 0$ such that for $T > T_0$,

$$\Phi_T(m) < \frac{\varepsilon}{K+1}$$

$$\text{for } 0 \leqslant m \leqslant K \quad (20)$$

Now by (18) – (20), for $T > T_0$ we have

$$\frac{1}{\mu_s(T,Q)} \sum_{n=-T}^{T} |\hat{c}_n^e| Q(n) = \sum_{m=0}^{K} |\lambda_1|^m \Phi_T(m) + \sum_{m=K+1}^{\infty} |\lambda_1|^m \Phi_T(m) + \sum_{m=0}^{K} |\lambda_1|^m \Phi_T(m) + \sum_{m=K+1}^{\infty} |\lambda_2|^m \Phi_T(m) \leqslant$$

$$2(K+1)\frac{\varepsilon}{K+1} + 2M_1 = 2(1+M_1)\varepsilon$$

which implies that $\{\hat{c}_n^e\}_{n\in\mathbf{Z}} \in PAP_0(\mathbf{R},Q)$ for $|\lambda_1| < 1$, $|\lambda_2| < 1$. Similarly, we can get that $\{\hat{c}_n^e\}_{n\in\mathbf{Z}} \in PAP_0(\mathbf{R},Q)$ for $|\lambda_1| < 1$, $|\lambda_2| > 1$ or $|\lambda_1| > 1$, $|\lambda_2| < 1$ or $|\lambda_1| > 1$, $|\lambda_2| > 1$.

(iii) Let $\varphi(s)$, $-1 \leqslant s \leqslant 0$, be a continuous function satisfying $\varphi(-1) = \hat{c}_{-1}$, $\varphi^1(0) = \hat{c}_0$. We define $x(t)$ step by step as follows:

$$x(t) = -px(t-1) + (\hat{c}_n + p\hat{c}_{n-1}) + [a_0\hat{c}_n + a_1\hat{c}_{n-1}](t-n) + \int_n^t f(s)\mathrm{d}s$$

$$(n \leqslant t < n+1)$$

with $x(s) = \varphi(s)$, $-1 \leqslant s \leqslant 0$.

It is easily seen that $x(t)$ is continuous, $x(n) = \hat{c}_n$, $n \in \mathbf{Z}$ and satisfies

$$[x(t) + px(t-1)]' = a_0(x[t]) + a_1x([t-1]) + f(t) \quad (21)$$

Let $w(t) = x(t) + px(t-1)$, we claim that $w(t) \in PAP_0(\mathbf{R},Q)$. Let $f = f^{ap} + f^e$, where $f^{ap} \in AP\omega(\mathbf{R})$, $f^e \in PAP_0(\mathbf{R},\rho)$, let

$$w^{ap}(t) = (\hat{c}_n^{ap} + p\hat{c}_{n-1}^{ap}) + [(a_0\hat{c}_n^{ap} + a_1\hat{c}_{n-1}^{ap})](t-n) + \int_n^t f^{ap}(s)\mathrm{d}s,$$

$$w^e(t) = w(t) - w^{ap}(t) = (\hat{c}_n^e + p\hat{c}_{n-1}^e) + [(a_0\hat{c}_n^e + a_1\hat{c}_{n-1}^e)](t-n) + \int_n^t f^e(s)\mathrm{d}s$$

We have

$$|w^{ap}(t+\omega) + w^{ap}(t)| = \Big|[\hat{c}_{n+\omega}^{ap} + p\hat{c}_{n+\omega-1}^{ap}] + [c_n^{ap} + pc_{n-1}^{ap}] + [a_0c_{n+\omega} + a_1c_{n+\omega-1}](t-n) + [a_0c_n + a_1c_{n-1}](t-n) + \int_{n+\omega}^{t+\omega} f^{ap}(s)\mathrm{d}s + \int_n^t f^{ap}(s)\mathrm{d}s\Big| =$$

$$(\hat{c}_{n+\omega}^{ap} + \hat{c}_n^{ap}) + p(\hat{c}_{n+\omega-1}^{ap} + \hat{c}_{n-1}^{ap}) + a_0(c_{n+\omega} + c_n)(t-n) + a_1[c_{n+\omega-1\,n} + c_{n-1}] + \int_n^t [f^{ap}(s+\omega) + f^{ap}(s)]\mathrm{d}s = 0$$

It follows from definition that $w^{ap}(t)$ is ω-anti-periodic. Denote

$$\eta_n = |\hat{c}_n^e| + |p\hat{c}_{n-1}^e| + |a_0\hat{c}_n^e| + |a_1\hat{c}_{n-1}^e|$$

Then $\{\eta_n\} \in PAP_0(\mathbf{R},Q)$. By an argument the same as the proof Lemma 4 we get that

$$\left\{\int_n^{n+1} |f^e(s)|\,\mathrm{d}s\right\} \in PAP_0(\mathbf{R},Q)$$

Meanwhile, it follows from Lemma 3 that there exists some $M > 0$ such that $\mu_s([T]+1, Q) \leqslant M\mu(T,\rho)$ for T large enough. Then for T large enough we have

$$\frac{1}{\mu(T,\rho)} \int_{-T}^{T} |w^e(t)\rho(t)|\,\mathrm{d}t \leqslant \frac{1}{\mu(T,\rho)} \sum_{n=-[T]-1}^{[T]} \int_n^{n+1} |w^e(t)|\rho(t)\mathrm{d}t \leqslant \frac{M}{\mu_s([T]+1,Q)} \cdot \sum_{n=-[T]-1}^{[T]+1} \int_n^{n+1} (\eta_n + \int_n^{n+1} |f^e(s)|\,\mathrm{d}s)\rho(t)\mathrm{d}t = \frac{M}{\mu_s([T]+1,Q)} \cdot$$

$$\sum_{n=-[T]-1}^{[T]} (\eta_n + \int_n^{n+1} |f^e(s)|\,\mathrm{d}s) Q(n) \to 0,$$

as $T \to \infty$

This implies that $w^e \in PAP_0(\mathbf{R},\rho)$, and hence $w \in PAP_\omega(\mathbf{R},\rho)$.

Next we express x in terms of w and then prove that $x \in PAP_\omega(\mathbf{R},\rho)$. From

$$w(t) = x(t) + px(t-1),$$

One has for all $n \in \mathbf{Z}^+$

$$x(t-n) = w(t-n) - px(t-n-1) \quad (22)$$

hence,

$$\forall n \in \mathbf{Z}^+, (-p)^n x(t-n) =$$

$$(-p)^n w(t-n)+(-p)^{n+1}x(t-n-1)$$

It follows

$$x(t)=\sum_{k=0}^{n}(-p)^k w(t-k)+(-p)^{n+1}x(t-n-1)$$

Since $|p|<1$ and x is bounded, then $(-p)^{n+1}x(t-n-1)\to 0$ as $n\to+\infty$, and

$$x(t)=\sum_{k=0}^{+\infty}(-p)^k w(t-k)$$

Conversely, if we put

$$x(t)=\sum_{k=0}^{+\infty}(-p)^k w(t-k)$$

x is well defined and w is bounded and $|p|<1$, x is bounded with $|x(t)|\leqslant \|w(t)\|_\infty/(1-|p|)$, moreover one has

$$x(t)+px(t-1)=\sum_{k=0}^{+\infty}(-p)^k w(t-k)-\sum_{k=0}^{+\infty}(-p)^{k+1}w(t-k-1)=w(t)$$

For $|p|<1$, rewriting (22) as

$$\left(\frac{-1}{p}\right)^n x(t+n)=\left(\frac{-1}{p}\right)^{n+1}w(t+n+1)+\frac{(-1)^n}{p^{n+1}}x(t-n-1)$$

we deduce in a similar manner that

$$x(t)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{p^{n+1}}\omega(t+n+1)$$

If $|p|<1$, given $\varepsilon>0$, there exists an integer $K>0$ such that

$$\sum_{n=K+1}^{\infty}|p|^n<\varepsilon \tag{23}$$

Let

$$x^{ap}(t)=\sum_{n=0}^{\infty}(-p)^n\omega^{ap}(t-n),$$

$$x^e(t)=x(t)-x^{ap}(t)=\sum_{n=0}^{\infty}(-p)^n\omega^e(t-n)$$

By a standard argument we can get that $x^{ap}\in AP_\omega(\mathbf{R})$. Since $PAP_0(\mathbf{R},\rho)$ with $\rho\in U_T$ is translation invariant, namely $\varphi\in PAP_0(\mathbf{R},\rho)$ and $s\in$ R imply that $\varphi(\cdot-s)\in PAP_0(\mathbf{R},\rho)$ (see [14, Lemma 4.1]), we get that $w^e(\cdot-n)\in PAP_0(\mathbf{R},\rho)$ for $n\in\mathbf{Z}^+$. So there exists $T_0>0$ such that for $T>T_0$,

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|w^e(t-n)|\rho(t)\mathrm{d}t<\frac{\varepsilon}{K+1},\quad \text{for } 0\leqslant n\leqslant K \tag{24}$$

Now by (23) and (24), for $T>T_0$,

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|x^e(t)\rho(t)|\,\mathrm{d}t\leqslant \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\sum_{n=0}^{\infty}|p|^n|w^e(t-n)|\rho(t)\mathrm{d}t=$$

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\sum_{n=0}^{K}|p|^n|w^e(t-n)|\rho(t)\mathrm{d}t+\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\sum_{n=K+1}^{\infty}|p|^n|w^e(t-n)|\rho(t)\mathrm{d}t\leqslant$$

$$\sum_{n=0}^{K}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|p|^n|w^e(t-n)|\rho(t)\mathrm{d}t+\varepsilon w^e\leqslant (K+1)\frac{\varepsilon}{K+1}+\varepsilon w^e=(1+w^e)\varepsilon$$

This implies that $x^e\in PAP_0(\mathbf{R},\rho)$, and $x\in PAP\omega(\mathbf{R},\rho)$.

If $|p|>1$, let

$$x^{ap}(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{n+1}}w^{ap}(t+n+1),$$

$$x^e(t)=x(t)-x^{ap}=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{n+1}}w^e(t+n+1)$$

Similarly to the above, we can prove that $x\in PAP_\omega(\mathbf{R},\rho)$.

(iv) If $\bar{x}(t)$ is another ρ-pseudo ω-anti-periodic solution of Eq. (1), then $\bar{x}(t)-x(t)$ is a ρ-pseudo ω-anti-periodic solution of the corresponding homogeneous equation. Thus, $\{\bar{x}(n)-x(n)\}_{n\in\mathbf{Z}}$ is a solution of the homogeneous difference equation (12). Hence, there exist k_1,k_2 such that

$$\bar{x}(n)-x(n)=k_1\lambda_1^n+k_2\lambda_2^n,n\in\mathbf{Z}$$

From the boundedness of the ρ-pseudo ω-anti-periodic function, it follows that $\bar{x}(n)-x(n)\equiv 0$, $n\in\mathbf{Z}$. At this time, $\bar{x}(t)-x(t)$ satisfies

$$|\bar{x}(t)-x(t)|+p[\bar{x}(t-1)-x(t-1)]=0,\quad \forall t\in\mathbf{R}$$

Clearly, we have

$$\bar{x}(t_0-n)-x(t_0-n)=\frac{(-1)^n}{p^n}[\bar{x}(t_0)-x(t_0)],t_0\in\mathbf{R},n\in\mathbf{Z}^+$$

From the boundedness of the ρ-pseudo ω-anti-periodic function, it follows that

$$\bar{x}(t)\equiv x(t),t\in\mathbf{R}$$

This means that the ρ-pseudo ω-anti-periodic solution of Eq. (1) is unique.

(v) If $f(t)$ is ρ-pseudo ω-anti-periodic and $\omega=n_0/m_0,n_0,m_0\in\mathbf{Z}^+$, then the sequence $\{h_n\}_{n\in\mathbf{Z}}$ is ρ-pseudo $m\omega$-anti-periodic bounded sequence, where $m=m_0$ if m_0 is odd, or $m=2m_0+1$ if m_0 is even. At this time, the sequence $\{\hat{c}_n\}$ defined by Eq. (17) is also ρ-pseudo $m\omega$-anti-periodic bounded sequence. Following step (iv), we can easily prove that Eq. (1)

possesses a ρ-pseudo $m\omega$-anti-periodic bounded solution. This completes the proof of Theorem 1.

Proof of Theorem 2 (i) It is easy to seen that the space $PAP_{\omega}(\mathbf{R},\rho)$ is a Banach space with supremum norm $\|\varphi\| = \sup_{t\in\mathbf{R}}|\varphi(t)|$. For any $\varphi \in PAP_{\omega}(\mathbf{R},\rho)$, using both (H2) and the composition of functions in $PAP_0(\mathbf{R},\rho)$ (see Diagana[19]), it follows that $g(t,\varphi(t),\varphi([t])) \in PAP_{\omega}(\mathbf{R},\rho)$.

We consider the following equation:

$$[x(t)+px(t-1)]' = a_0(x[t]) + a_1x([t-1]) + g(t,\varphi(t),\varphi([t])) \tag{25}$$

From Theorem 1, it follows that for any $\varphi \in PAP_{\omega}(\mathbf{R},\rho)$, Eq. (25) has a unique weighted pseudo-anti-ω-periodic solution, denote by $J\varphi$. Thus, we obtain a mapping $J:\varphi \to x_{\varphi}$, it follows that J is a mapping from $PAP_{\omega}(\mathbf{R},\rho)$ into itself. For any φ, $\psi \in PAP_{\omega}(\mathbf{R},\rho)$, $J\varphi - J\psi$ satisfies the following equation:

$$[z(t)+pz(t-1)]' = a_0(z[t]) + a_1z([t-1]) + g(t,\varphi(t),\varphi([t])) - g(t,\psi(t),\psi([t]))$$

Since $J\varphi$ and $J\psi$ are ρ-pseudo ω-anti-periodic, there exists a constant $B>0$ such that $|J\varphi|, |J\psi| \leqslant B$.

Setting $\tilde{c}_n = (J\varphi)(n) - (J\psi)(n)$, then we have

$$\tilde{c}_{n+1} + (p-1-a_0)\tilde{c}_n + (-p-a_1)\tilde{c}_{n-1} = H_n$$

where

$$H_n = \int_n^{n+1}[g(s,\varphi(s),\varphi([s])) - g(s,\psi(s),\psi([s]))]\mathrm{d}s$$

From the proof of the Theorem 1, we know that there exist k_1^*, k_2^* such that

$$\tilde{c}_n = k_1^*\sum_{m\leqslant n-1}\lambda_1^{n-(m+1)}H_m + k_2^*\sum_{m\leqslant n-1}\lambda_2^{n-(m+1)}H_m$$

This implies that there exists $K_0>0$, such that

$$|(J\varphi)(n) - (J\psi)(n)| = K_0\sup_{n\in\mathbf{Z}}|H_n| \leqslant K_0\eta|\varphi-\psi|, \quad \forall n\in\mathbf{Z}$$

Let

$$H(t) = (\tilde{c}_n + p\tilde{c}_{n-1}) + [a_0\tilde{c}_n + a_1\tilde{c}_{n-1}](t-n) + \int_n^t[g(s,\varphi(s),\varphi([s])) - g(s,\psi(s),\psi([s]))]\mathrm{d}s, \quad (n\leqslant t<n+1)$$

Thus there exists $K_1>0$ such that

$$|H(t)| \leqslant K_1\eta|\varphi-\psi|$$

We easily conclude that

$$(J\varphi)(t) - (J\psi)(t) + p[(J\varphi)(t-1) - (J\psi)(t-1)] = H(t)$$

We typically consider the case when $|p|<1$. Using Lemma 5, we have

$$|(J\varphi)(t) - (J\psi)(t)| \leqslant e^{-a(t-t_0)}\lim_{-1\leqslant\theta\leqslant0}|(J\varphi)(t_0+\theta) - (J\psi)(t_0+\theta)| + b\sup_{t_0\leqslant u\leqslant t}|H(u)| \leqslant e^{-a(t-t_0)}2B + bK_1\eta|\varphi-\psi|, \ t\geqslant t_0$$

where $a = \log(1/|p|)$, $b = 1/(1-|p|)$. Setting $t_0 \to \infty$, we obtain

$$|J\varphi - J\psi| \leqslant bK_1\eta|\varphi-\psi|$$

Hence, there exists $\eta^*>0$, such that if $0\leqslant\eta<\eta^*$, $J:PAP_{\omega}(\mathbf{R},\rho) \to PAP_{\omega}(\mathbf{R},\rho)$ is contracting mapping. This implies that there exists $\varphi \in PAP_{\omega}(\mathbf{R},\rho)$ such that $J\varphi = \varphi$ that is, Eq. (1) has a unique ρ-pseudo ω-anti-periodic solution.

(ii) If $\omega = (n_0/m_0)(n_0, m_0 \in \mathbf{Z}^+)$ and g is ρ-pseudo ω-anti-periodic in t, then $g(t,\varphi(t),\varphi([t]))$ is a ρ-pseudo m_0 ω-anti-periodic function, for any $\varphi \in PAP_{m_0\omega}(\mathbf{R},\rho)$. At this time, it follows that Eq. (25) has a unique ρ-pseudo $m_0\ \omega$-anti-periodic solution $J\varphi$ by using Theorem 1. Similarly, we know that there exists $\eta^*>0$ such that if $0\leqslant\eta<\eta^*$, Eq. (1) has a unique ρ-pseudo $m_0\ \omega$-anti-periodic solution. This completes the proof of Theorem 2.

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